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Synchronized traffic flow from a modified Lighthill-Whitman model

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A simple macroscopic argument leading to a diffusively corrected form of the classical kinematic-wave (Lighthill-Whitham) model of the flow of vehicular traffic is described. An example of a diffusively corrected kinematic-wave model displays a diffusion coefficient that is negative, for sufficiently large densities. It is shown that such a diffusively corrected kinematic-wave model is capable of reproducing elements of the synchronized flow reported by Kerner and Rehborn.

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The author and Sopasakis have recently [1] shown that: (i) the zero-order Chapman-Enskog approximation to the classical Prigogine-Herman [2] kinetic equation of vehicular traffic is a Lighthill-Whitham model, which consists of the continuity equation,

$$\frac{\partial \rho}{\partial t}+\frac{\partial q}{\partial x}=0,$$

and a classical traffic stream model $q=Q_0(\rho)$; (ii) the corresponding first-order Chapman-Enskog approximation, and therefore presumably the proper traffic-theoretic analog of the Navier-Stokes equations of fluid dynamics, is a diffusively corrected Lighthill-Whitham model. Such a model consists of the continuity equation and a diffusively corrected traffic stream model,

$$q(x,t)=Q_0(\rho(x,t))-D(\rho(x,t))\frac{\partial \rho}{\partial x}(x,t). \tag{1}$$

Lighthill and Whitham themselves [3] expressed considerable skepticism about the validity of classical traffic stream models. They further suggested an extension that contained a diffusive term, which was considered as representing anticipation. Schochet [4] showed that the entropy weak solutions of the Lighthill-Whitham model are the limits, as the constant in a diffusive coefficient of the form const$\times\rho^{-1}$ tends to zero, of the solutions of the corresponding diffusively corrected Lighthill-Whitham model.

If an additive diffusive correction to the traffic stream model improves on Lighthill-Whitham models, then it should be possible to establish a simple phenomenological basis for such a correction. This would be analogous to the phenomenological basis for the Navier-Stokes equations, as asserted well before the *tour de force* for atomistic theory comprised by the demonstrations of Chapman and of Enskog that the Navier-Stokes equations follow from the Boltzmann equation, complete with prescriptions for the diffusion coefficient and viscosity, in terms of an intermolecular force law. Toward establishing such a phenomenological basis, note that if drivers intend to act so as to adjust their speed to the local density of vehicles at any given position and time, then that intention inevitably is somewhat frustrated by the existence of some *reaction time* τ, representing a delay in their response to events. I assume that drivers compensate for this delay by adjusting to the density seen at some *anticipation distance* L ahead of their current position. In view of the reaction time and the anticipation length, the actual (mean) speed at position x and time t will then be

$$v(x,t)=V(\rho(x+L-V\tau,t-\tau)),$$

where $V(\rho)$ is the (mean) desired speed at density ρ.

If the right-hand side is expanded to first order in τ and L, and higher-order terms are ignored, then after a bit of algebra results are

$$q:=\rho v=Q_0(\rho)+\rho\{LV'(\rho)+\tau\rho[V'(\rho)]^2\}\frac{\partial \rho}{\partial x}.$$

This has precisely the form of Eq. (1), with the diffusion coefficient given by

$$D(\rho)=-L\rho V'(\rho)-\tau\rho^2[V'(\rho)]^2. \tag{2}$$

This equation displays the diffusion coefficient as the *difference* of two non-negative terms, as V' is nonpositive. One would normally expect D to be non-negative, as this is necessary in order for the corresponding initial-value problem to be well-posed. (For the first-order Chapman-Enskog approximation to the Prigogine-Herman kinetic equation, which is valid only up to some ''critical'' density, this result is proved in [1].) Presumably this expectation is reflected in the tendency of drivers to anticipate so as to more than compensate for the reaction time, so that the first term in Eq. (2) would normally be larger in magnitude than that associated with the reaction time. However, it is certainly conceivable that there are situations in which reaction delay dominates anticipation, so that the resulting diffusion coefficient is negative.

In order to explore this issue further, as well as to obtain an order-of-magnitude estimate of the diffusion coefficient, consider the Dick [5] modification of the traffic stream model of Greenberg [6],

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$$V(\rho)=\min\{v_{max},C\ln(\rho_{max}/\rho)\},$$

with the freeflow speed $v_{max}=70$ miles per hour, the jam density $\rho_{max}=220$ vehicles per lane-mile, and the ''free'' parameter $C=10e$ (miles per hour). The corresponding value of V' is

$$V'(\rho)=\begin{cases}0, & \rho<e^{-7/e}\rho_{max}:=\hat{\rho}\approx 0.076\rho_{max}\\ -10e/\rho\approx -27.2/\rho, & \rho>e^{-7/e}\rho_{max}\approx 0.076\rho_{max}\end{cases}\quad \text{miles}^2\text{ per hour},$$

with densities in units of miles per hour. For purposes of an order-of-magnitude discussion I shall take the reaction time as $\tau=2$ sec$=0.000\,555\,5\ldots$ h, which is a typical time for driver reaction (e.g., [7]). Similarly, I shall take the anticipation length as $L=v^2/15\,800$, which is the distance (in miles) required to decelerate to a full stop from v miles per hour, at a relatively comfortable deceleration of about $0.1g$. The corresponding diffusion coefficient is plotted, as a function of density, in Fig. 1.

For this situation, the diffusion coefficient is positive for $\rho<\rho_0\approx 124.6$ vehicles per mile, but is (slightly) negative for $\rho>\rho_0$. Therefore, the corresponding diffusively corrected Lighthill-Whitham model may be mildly unstable for densities above ρ_0. We may expect this instability effect for even smaller densities, and perhaps at a more severe level, if the anticipation length were not be taken as large as assumed above. This might arise, for example, at roadway locations providing limited sight distance ahead, or near entrance ramps, at which drivers ability to anticipate conditions ahead is reduced by the necessity to accommodate merging vehicles. Negative diffusion coefficients are known to be associated with a number of interesting phenomena [8].

I now turn to the issue of the consistency of synchronized flow [9] with the diffusively corrected Lighthill-Whitham model just described. For this purpose I understand ''synchronized flow'' as traffic flow in which speeds are synchronized across lanes, and those speeds are too low to be considered free flow, but too high to be considered a traffic jam. In order to reproduce theoretically this aspect of traffic flow, one needs some model that provides speeds that are specific to individual lanes. Several such models exist (e.g., [10]). However, any realistic lane-specific model should reflect the natural desire (and ability) of drivers in a slower moving lane to transfer to a faster moving lane, and that effect should tend to equilibrate the speed across lanes, as suggested in [11]. Therefore, this aspect of synchronized flow does not seem so challenging theoretically. The further observation that ''synchronized traffic flow covered on the flux-density plane two-dimensional regions'' is rather a different matter, as elaborated further below. However, that issue can be addressed without lane-specific models, because it can be viewed as relating to the behavior of traffic *after* the equilibration of speeds across lanes has been achieved, so that there is little incentive for lane switching to occur.

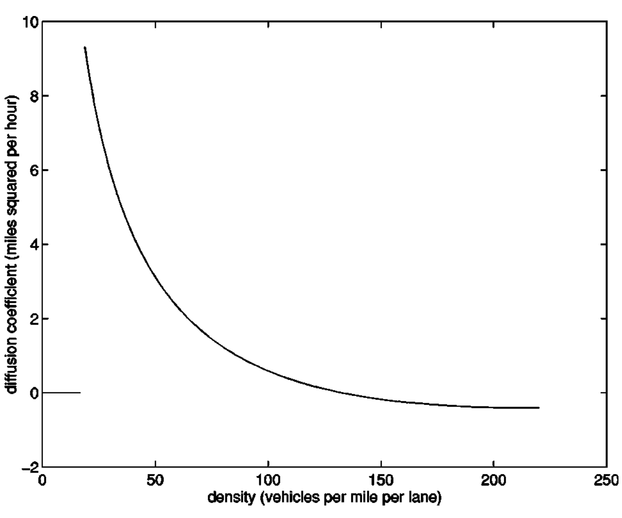

FIG. 1. Dependence of the diffusion coefficient upon density.

In addition, Kerner and Rehborn [9] describe types (i), (ii), and (iii) of synchronized traffic flow, and assert that ''each of these three types of states of synchronized traffic flow covered on the flux-density plane two-dimensional regions.'' The diffusive correction to the flow permits this possibility, in contrast to the kinematic-wave model itself, which predicts that all traffic states must lie along the curve $q=Q_0(c)$ (i.e., the ''static'' traffic stream model) in the density/flow plane. But can this possibility be realized in a manner that is consistent with the other essential observed properties of synchronized flow, as embodied in their defining descriptions? An outline of an affirmative answer to this question will now be provided analytically, for the case of type (ii) synchronized flow. A similar demonstration can be provided for type (i) synchronized flow.

Type (ii) synchronized flow. Type (ii) synchronized flow is described [9] as ''states, where the average speed was nearly a stationary one during a relatively long time interval, but the flux, i.e., the density, noticably changed during this time interval.'' Accordingly, I seek solutions of the diffusively corrected Lighthill-Whitham model that further satisfy $q=w\rho$, where w is independent of both x and t. (The defining statement quoted above requires only that w be independent of time; the literature seems silent on spatial homogeneity, although the terminology ''homogeneous-in-speed'' [12] is strongly suggestive.) With the continuity equation, this requires solutions of type (ii) have densities that satisfy $\rho(x,t)=c(x-wt)$, where c is an arbitrary function. The diffusively corrected ''dynamic'' traffic stream model then yields the equation

$$D(c)\frac{dc}{dx_0}=Q_0(c(x_0))-wc(x_0), \qquad (3)$$

where x_0 is the initial location of the ''wave'' moving along the characteristic $x = x_0 + wt$ (and w is the associated ''wave speed'').

Equation (3) is an ordinary differential equation that can, in principle, be solved for an initial-density profile, $\rho(x_0,0) = c(x_0)$, which will possibly evolve into a flow of type (ii). It is possible to give an exhaustive analysis of the behavior of the solutions of this equation. Here I shall only note a few cases, in order to illustrate that even this subclass of type (ii) solutions occupies a significant two-dimensional region of the density/flow plane.

Let ρ_w denote the root of $Q(\rho) = w\rho$.

ii-a. If $Q_0(\rho_0)/\rho_0 < w < v_{max}$, then the solution corresponding to an initial value $\rho(x_1)$ such that $\hat{\rho} < \rho(x_1) < \rho_w$ approaches ρ_w, asymptotically as $x \to \infty$. [This type of solution appears to correspond to the interior of what would be seen as a shock in the Lighthill-Whitham theory. It joins smoothly to (stable) downstream flow at density ρ_w and speed w.]

ii-b. If $Q_0(\rho_0)/\rho_0 < w < v_{max}$, then the solution corresponding to an initial value $\rho(x_1)$ such that $\rho_w < \rho(x_1) < \rho_0$ approaches ρ_w, asymptotically as $x \to \infty$. (A solution of this type appears to have some similarity to what is frequently termed ''queue discharge.'' As in the preceding case, this flow joins smoothly to a stable downstream flow at density ρ_w and speed w.

ii-c. If $w < Q_0(\rho_0)/\rho_0$, then the solution corresponding to any initial value $\rho(x_1)$ such that $\rho_0 < \rho(x_1) < \rho_w$ approaches ρ_0 in finite length. [This is our final solution that lies in the region where the diffusion coefficient is negative. The fact that drivers are maintaining speed w here means they are reacting (by increasing their speeds) too slowly to the decreasing downstream density to maintain flow at the (now unstable) equilibrium corresponding to density ρ_w and speed w. This flow ultimately must undergo a discontinuity at or before the point $\rho(x) = \rho_0$, where the second-order parabolic diffusively corrected Lighthill-Whitham model degenerates into a first-order hyperbolic equation, in order to provide a connection to any downstream flow.]

ii-d. If $w < Q_0(\rho_0)/\rho_0$, then the solution corresponding to any initial value $\rho(x_1)$ such that $\max\{\rho_w, \rho_0\} < \rho(x_1) < \rho_{max}$ approaches ρ_{max} in finite length. (This corresponds to flow in a region that is controlled by a downstream jam; that is, this is flow in a transition to a jam. In maintaining speed w, drivers are going faster than predicted by the static traffic stream model, and all the more faster than warranted by downstream conditions. This reflects the fact that the diffusion coefficient is negative, because reaction dominates anticipation.)

The regions of the density/flow plane that are covered by these four cases of type (ii) synchronized flow are shown, in relation to the ''static'' traffic flow model, in Fig. 2. This figure illustrates that these classes alone of type (ii) synchronized-flow solutions of the diffusively corrected Lighthill-Whitham model cover a significant two-dimensional region in the density/flow plane. Other such type (ii) synchronized-flow solutions of the diffusively corrected Lighthill-Whitham model [i.e., other solutions of the differential equation (3)] provide even further two-dimensional coverage of the density/flow plane.

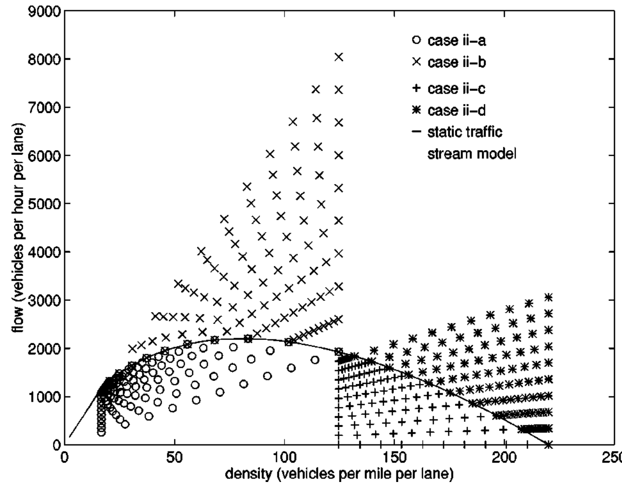

FIG. 2. The regions of the density/flow plane covered by the enumerated instances of type (ii) synchronized flow.

The most striking, and apparently unrealistic, feature of Fig. 2 is the rather large flows predicted for type ii-b synchronized flow. The largest flows occur when w is slightly less than v_{max} and ρ is slightly less than ρ_0. It is hard to conceive of how this situation could initially occur, but the theory predicts that if it did, and the speed were w along a density profile satisfying Eq. (3), then that speed would be maintained along that profile as it moves at speed w. However, along the density profile corresponding to a speed of $w = 65$ miles per hour, the density would drop from 120 vehicles per mile to approximately 33 vehicles per mile, within a length of about 0.1 miles. Therefore, the theory also predicts that such flows are nonsustainable, and in fact unobservable for all practical purposes. There are other practically nonexistent flows among the synchronized flows of types (i) and (ii) that are revealed by the approach demonstrated here. This fact notwithstanding, there remain sufficient of these flows (e.g., type ii-b flow corresponding to smaller values of the speed) to occupy a significant region of the density/flow plane. The larger point is that not all instances of the types of synchronized flow shown here to result from the diffusively corrected kinematic-wave model are equally likely to be observed.

Conclusions. A diffusive correction to the Lighthill-Whitham kinematic-wave model is readily justified, on the basis of the familiar phenomena of driver anticipation and reaction time. Yet this simple extension of the Lighthill-Whitham model can provide significantly different predictions of traffic flow, especially in regions of large gradients in the concentration, or in which the diffusion coefficient takes on a negative value. Here it has been shown that such a diffusive correction can reproduce significant elements of the synchronized flow described by Kerner and Rehborn [9].

It would be most interesting to explore the extent to which a diffusively corrected kinematic-wave model is able to reproduce other observed traffic-flow patterns and phenomena, including ''wide traffic jams'' [13], specific properties of phase transitions [14], the existence of a line dividing the density/flow into ''stable'' and ''metastable'' regions with regard to perturbations leading to phase transitions [15], and the oscillations noted by Koshi, Iwasaki and Ohkura [11].

Such explorations are given further impetus by the recent independent confirmation by Neubert *et al.* [16] of the three phases of traffic flow suggested by Kerner *et al.* This is important, because Banks [17] had previously noted that some of the observations of Kerner *et al.* could be attributed simply to statistical fluctuations in the data. This now seems rather less likely, although perhaps it cannot yet be ruled out altogether.

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